Παράδειγμα ($y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$)

$y''(x) = \alpha(x)\cdot y'(x) + x^2 \cdot y(x) + y(x)\cdot y'(x) + y^2(x)$

$f(x, y_1, y_2) = \alpha(x)\, y_2 + x^2 y_1 + y_1 y_2 + y_1^2$, $y_2 = y'$, $y_1 = y$

Διανυσματικές διαφορικές εξισώσεις:

$$\begin{cases} y_1'(x) = 2x^2 + 3y_1(x) + x^3 \cdot y_2^2(x) \\ y_2'(x) = \sin(y_2(x)) + 3x^2 \cdot y_1^2(x)\cdot y_2(x) \end{cases}$$

$$\begin{cases} y_1'(x) = f_1(x, y_1, y_2) = 2x^2 + 3y_1 + x^3 y_2^2 \\ y_2'(x) = f_2(x, y_1, y_2) = \sin y_2 + 3x^2 \cdot y_1^2 \cdot y_2 \end{cases} \Rightarrow$$

$\rightarrow \bar{y}'(x) = \bar{f}(x, \bar{y})$ όπου $\bar{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$

και έτσι $\bar{f}(x, \bar{y}) = \begin{bmatrix} f_1(x, y_1, y_2) \\ f_2(x, \bar{y}) \end{bmatrix} = \begin{bmatrix} f_1(x, \bar{y}) \\ f_2(x, \bar{y}) \end{bmatrix}$.

Παράδειγμα 1ο: Να γραφούν διανυσματικά οι εξισώσεις:

i) $y_1' = y_2 + 1$ | $y_1(0) = 1$

$y_2' = x + y_1^3$ | $y_2(0) = -2$

Λύση

$$\bar{y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 1 + y_2 \\ x + y_1^3 \end{bmatrix} = \begin{bmatrix} f_1(x, \bar{y}) \\ f_2(x, \bar{y}) \end{bmatrix} = \bar{f}(x, \bar{y})$$

όπου

$\bar{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Παράδειγμα 2ο:

$$\begin{cases} y_1' = y_2 + y_3 \\ y_2' = e^x y_1 + y_2 - (\sin x) y_3 \\ y_3' = y_1 \end{cases}$$

$$\bar{y}'(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 1 \\ e^x & 1 & -\sin x \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$\Rightarrow \bar{y}'(x) = A(x)\, \bar{y}$

Παράδειγμα 3ο:

$$\begin{cases} y_1'' = 5y_1 + e^x \cdot y_2 \\ y_2' = x y_1 + x^2 y_2 + \sin x \end{cases} \qquad y_1(0) = 1, \; y_1'(0) = -7, \; y_2'(0) = -1$$

Λύση

Θέτω $y_1 = u_1$, $y_1' = u_2$, $y_2 = u_3$

Άρα,

$$\begin{cases} u_2' = 5u_1 + e^x u_3 \\ u_1' = y_1' = u_2 \\ u_3' = x u_1 + x^2 u_3 + \sin x \end{cases} \rightsquigarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & e^x \\ x & 0 & x^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \sin x \end{bmatrix} \Rightarrow \bar{u}'(x) = A(x)\bar{u} + \bar{b}(x)$$

Μετατροπή σε διανυσματική Α' τάξεως.

$$y^{(n)}(x) = f(x, \underset{y_1}{y}, \underset{y_2}{y'}, \dots, \underset{y_n}{y^{(n-1)}})$$

$$\begin{aligned} y_1 &= y \\ y_2 &= y' = y_1' \\ &\vdots \\ y_{n-1} &= y^{(n-2)} \\ y_n &= y^{(n-1)} \end{aligned} \qquad \bar{y}'(x) = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix}' = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(x, y_1, \dots, y_n) \end{bmatrix}$$

$$y'' = a(x)\, y + b(x)\, y' + c(x)$$

όπου $f(x, y, y') = a \cdot y + b(y') + c$

Θέτω

$$\begin{aligned} y &= y_1 \\ y' &= y_2 \end{aligned} \quad \Bigg\| \quad \bar{y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ a y_1 + b y_2 + c \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Παράδειγμα 1:

$$(x^2+1)\,y''' - 5x\,y' + x^2\cdot y = e^x \quad , \quad y(2)=0 \;,\; y''(2)=-3,\; y'(2)=1$$

ΛΥΣΗ

$$y''' = \frac{5x}{x^2+1}\,y' - \frac{x^2}{x^2+1}\,y + \frac{e^x}{x^2+1}$$

$$f(x, y, y') : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$$

$y_1 = y$
$y_2 = y'$
$y_3 = y''$